DOCILITY AT INFINITY
AND COMPACTIFICATIONS OF ANR'S

BY

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ABSTRACT. Various conditions of contractibility and extensibility at ∞ for locally compact metric spaces are studied. These are shown to be equivalent if the space under consideration is an absolute neighborhood retract (ANR) and an ANR satisfying them is called *docile at* ∞. Docility at ∞ is invariant under proper homotopy domination. The ANR X is docile at ∞ if and only if FX (the Freudenthal compactification of X) is an ANR and $FX - X$ is unstable in FX; the inclusion of X into FX is a homotopy equivalence.

1. **Introduction.** In recent years there has been considerable interest in the geometry of noncompact spaces, particularly "at ∞". In this paper we consider various conditions of contractibility and extensibility at ∞ (alternatively, "at the ends") for locally compact metric spaces. These conditions are shown, in §3, to be equivalent provided the space in question is an absolute neighborhood retract, and an absolute neighborhood retract satisfying them will be called *docile* at ∞. Docility at ∞ is shown to be invariant under proper homotopy domination.

In §4 it is shown that if X is an absolute neighborhood retract which is docile at ∞, then FX, the Freudenthal compactification of X, is an absolute neighborhood retract, and that EX, the end-set of X, is unstable in FX. The converse is also shown to hold, and it is shown that the inclusion of X into FX is a homotopy equivalence.

2. **Definitions, notations, and other preliminaries.** Although not always required for our results, it will be convenient to adopt the convention for this paper that *all spaces henceforth considered shall be locally compact separable and metrizable.*

We use I to denote the interval $[0, 1]$ and 1_X to denote the identity function on the set X. The notations $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Fr} A$ are used for the closure, interior, and frontier of the set A relative to some (understood from context) am-

bient space. $X \in$ ANR (AR) means that X is an absolute neighborhood retract (absolute retract) for metric spaces.

Map will always mean continuous function. The map $f: X \longrightarrow Y$ is *proper* if $f^{-1}(C)$ is compact whenever C is a compact subset of Y. If Y_0, Y_1, and Y_2 are subspaces of the space Y, then maps $f_1: X \longrightarrow Y_1$ and $f_2: X \longrightarrow Y_2$ are *homotopic in Y_0* if there exists a map $F: X \times I \longrightarrow Y_0$ such that $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for all $x \in X$; similarly f_1 and f_2 are *properly homotopic in Y_0* if the map F can be chosen so as to be proper. If the maps f, $g: X \longrightarrow Y$ are homotopic (properly homotopic) in Y, we say simply that f and g are *homotopic* (*properly homotopic*) and write $f \simeq g$ ($f \simeq_p g$). The map $f: X \longrightarrow Y_0 \subset Y$ is *null-homotopic in Y* if f is homotopic in Y to a constant map; Y_0 is *contractible in Y* if the inclusion of Y_0 into Y is nullhomotopic.

Let Σ denote the class of separable locally compact metrizable spaces having compact quasi-component space. For example, if X is a locally compact polyhedron, then $X \in \Sigma$ if and only if X has only finitely many components. Then, if $X \in \Sigma$, the Freudenthal compactification [5] of X, denoted FX, is metrizable. We denote $FX - X$ by EX. If X, $Y \in \Sigma$, and $f: X \longrightarrow Y$ is a proper map, then f has a unique extension to a map $Ff: (FX, EX) \longrightarrow (FY, EY)$ [2, Lemma 4.2] or [8, Theorem 3]; such a map is *end-preserving* if $Ff | EX$ is injective. An alternative treatment of FX appears in [1].

3. Contractibility and extensibility at ∞. In this section we consider various contractibility and extensibility conditions at ∞ for the space X. These are shown to be equivalent if $X \in$ ANR, and they are also shown to be hereditary proper homotopy invariants in this case. These conditions describe a sort of maximal homotopy triviality at the ends of X.

(3.1) DEFINITION. The space X is *contractible at ∞* (X is $C(E)$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that each component of $X - B$ is contractible in $X - A$.

Our next two results relate the above concept to the idea of the local contractibility of a set in relation to an ambient space. Definitions may be found in [4].

(3.2) THEOREM. *If $X \in \Sigma$, then X is $C(E)$ if X, regarded as a subspace of FX, is LC at e for every $e \in EX$. The converse holds provided X is locally connected modulo a compact set.*

PROOF. Suppose first that X is LC at e for all $e \in EX$. Let A be a compact subset of X. Then, since $FX - A$ is an open subset of FX containing EX, for each $e \in EX$ there exists an open set V_e of FX such that $e \in V_e \subset FX - A$, Fr V_e is compact, and $X \cap V_e$ is contractible in $X \cap (FX - A) = X - A$. Let B be a compact subset of X such that each component of $X - B$ lies in one of the

sets V_e, $e \in EX$. Then each component of $X - B$ is contractible in $X - A$, thereby showing that X is $C(E)$.

Now suppose that X is $C(E)$ and locally connected modulo a compact set. Let $e \in EX$ and let U be a neighborhood of e in FX. Then there exist pairwise disjoint open sets $U_1, U_2, \ldots, U_n$ in FX such that $EX \subset \bigcup_{i=1}^n U_i$ and $e \in U_1 \subset U$. Then $A = FX - \bigcup_{i=1}^n U_i$ is a compact subset of X. Since X is $C(E)$, there exists a compact set $B \subset X$ such that each component of $X - B$ is contractible in $X - A$. Since X is locally connected modulo a compact set, there exists a connected open set V of FX such that $e \in V \subset FX - B$ and Fr $V \subset X$. Let $V' = V \cap X$. Then V' is a connected subset of $X - B$ and thus V' lies in some component of $X - B$. It follows that V' is contractible in $X - A = \bigcup_{i=1}^n (U_i \cap X)$. This implies that V' is contractible in $U_1 \cap X \subset U \cap X$. This shows that X is LC at e, and the proof is complete.

The hypothesis that X be locally connected modulo a compact set is necessary in the second part of the above proof. An example to show this is given by $X = \bigcup_{i=0}^\infty A_i \subset R^2$, where

$$A_0 = \{(x, 0) | 0 \leqslant x \leqslant \tfrac{1}{2}\}, \quad A_1 = \{(0, y) | y \geqslant 0\}, \quad \text{and}$$

$$A_i = \{(1/i, y) | y \geqslant 0\} \quad \text{for } i = 2, 3, \ldots.$$

Now, if A is a subset of the metric space X and Cl A is compact, then A is ULC if and only if A is LC rel X [4, p. 615]. Thus Theorem 3.2 immediately yields the following. In its statement, a metric on X *compatible* with FX is the restriction to X of a metric on FX.

(3.3) COROLLARY. *Let $X \in \Sigma$ be equipped with a metric compatible with FX. Then X is $C(E)$ and LC if and only if X is ULC.*

The following shows that the property of a space being $C(E)$ is invariant under proper homotopy domination.

(3.4) THEOREM. *If $X \leqslant_p Y$ and Y is $C(E)$, then X is $C(E)$.*

PROOF. Suppose Y is $C(E)$ and $X \leqslant_p Y$. Let $h: X \longrightarrow Y$, $g: Y \longrightarrow X$, and $F: X \times I \longrightarrow X$ be proper maps such that $F(x, 0) = x$ and $F(x, 1) = gh(x)$ for all $x \in X$. If A is a compact subset of X then, letting $A' = g^{-1}(A)$, there exists a compact set $B' \subset Y$ such that each component of $Y - B'$ is contractible in $Y - A'$. Let B be a compact subset of X such that $F^{-1}(A) \subset B \times I$ and $h^{-1}(B') \subset B$. In order to show that X is $C(E)$, it will suffice to show that each component of $X - B$ is contractible in $X - A$.

Let P be a component of $X - B$. Then $h(P)$ lies in a component P' of $Y - B'$. By our choice of B', there exists a point $p_0' \in Y$ and a map $G: P' \times I$

$\longrightarrow Y$ such that $G(P' \times I) \subset Y - A'$, $G(p', 0) = p'$, and $G(p', 1) = p_0'$ for all $p' \in P'$. Define $H: P \times I \longrightarrow X$ by

$$H(p, t) = \begin{cases} F(p, 2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ g(G(h(p), 2t - 1)) & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

for all $p \in P$ and $t \in I$. Then H is easily seen to be a contraction of P in $X - A$, thereby completing the proof.

(3.5) DEFINITION. The space X is *C-connected at* ∞ (X is $C - C(E)$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that if C_0 is a compactum and $f: C_0 \longrightarrow X - B$ is a map carrying C_0 into a component of $X - B$, then f is nullhomotopic in $X - A$.

(3.6) DEFINITION. The space X is *K-connected at* ∞ (X is $K - C(E)$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that if K_0 is a compact polyhedron and $f: K_0 \longrightarrow X - B$ is a map carrying K_0 into a component of $X - B$, then f is nullhomotopic in $X - A$.

Obviously X is $K - C(E)$ if it is $C - C(E)$. That the two notions differ in general can be seen from the following example. Let W be a "Warsaw circle" and let w_0 be a point of W at which W is locally connected. Let X be the space obtained by attaching to the nonnegative reals a homeomorphic copy W_i of W at each nonnegative integer i, identifying the integer i with the point in W_i corresponding to w_0. Since W_i is not contractible in X for $i = 0, 1, \ldots$, it follows that X is not $C - C(E)$. However, using the fact that the continuous image of a compact polyhedron is locally connected, it can easily be verified that X is $K - C(E)$. By the following, such disparity cannot occur if the space under consideration is an absolute neighborhood retract.

(3.7) THEOREM. *If $X \in ANR$, then X is $C - C(E)$ if and only if X is $K - C(E)$.*

PROOF. It is necessary to show that if $X \in ANR$ and X is $K - C(E)$, then X is $C - C(E)$. To this end, let A be a compact subset of X, and let B be a compact subset of X such that any map of a compact polyhedron into a component of $X - B$ is nullhomotopic in $X - A$. Let X_0 be a component of $X - B$. Then X_0 is open in X. Hence $X_0 \in ANR$. Now, let C_0 be a compactum and $f: C_0 \longrightarrow X_0$ a map. We shall show that f is nullhomotopic in $X - A$, thereby showing that X is $C - C(E)$.

Since $X_0 \in ANR$, there exists a "bridge" for f [6, Chapter IV, §8]. More precisely, there exists an open cover α of C_0 such that if β is any locally finite open refinement of α, if N_β is the nerve of β, and if $\phi_\beta: C_0 \longrightarrow N_\beta$ is a canonical map, then there exists a map $g_\beta: N_\beta \longrightarrow X_0$ such that $f \simeq g_\beta \phi_\beta$. Since C_0 is com-

pact, we may choose such a β which is a finite refinement of α, so that N_β is a compact polyhedron. Then, by our choice of B, g_β is nullhomotopic in $X - A$. This implies that f is nullhomotopic in $X - A$, thereby completing the proof.

(3.8) DEFINITION. The space X is an *absolute extensor at ∞ for compacta* $(X \in AEC(E))$ if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that if $D_0 \subset C_0$ are compacta and $f: D_0 \longrightarrow X - B$ is a map carrying D_0 into a component of $X - B$, then there exists a map $f^*: C_0 \longrightarrow X - A$ such that $f^*(d) = f(d)$ for all $d \in D_0$.

Clearly X is $C - C(E)$ if $X \in AEC(E)$. But the converse does not generally hold, as seen from the following example. Let F denote the cone on the set $\{0, 1/2, 1/3, 1/4, \ldots\} \subset I$ (a fan) with vertex v. Let X be the space obtained by attaching to R^+, the nonnegative reals, a homeomorphic copy F_i of F at each nonnegative integer i, identifying the integer i with the point $v_i \in F_i$ corresponding to v. It is easy to see that R^+ is a strong deformation retract of X (under a homotopy which contracts each F_i to $\{v_i\}$), and that this implies that X is $C - C(E)$. However, $X \notin AEC(E)$, for if $D_0 = \{0, 1/2, 1/3, 1/4, \ldots\} \subset I = C_0$ and i is a nonnegative integer, then there is a map of D_0 into F_i (corresponding to the inclusion of D_0 into F) which cannot be extended to a map of C_0 into X. The following result shows that this sort of behavior cannot occur if $X \in$ ANR.

(3.9) THEOREM. *If $X \in ANR$, then X is $C - C(E)$ if and only if $X \in AEC(E)$.*

PROOF. It is necessary to show that if $X \in$ ANR and X is $C - C(E)$, then $X \in AEC(E)$. For this purpose, let A be a compact subset of X, and let B be a compact subset of X such that if C' is a compactum and $f': C' \longrightarrow X - B$ is a map carrying C' into a component of $X - B$, then f' is nullhomotopic in $X - A$. Let X_0 be a component of $X - B$, let $D_0 \subset C_0$ be compacta, and let $f: D_0 \longrightarrow X_0$ be a map. To show that $X \in AEC(E)$, it will suffice to show that there exists a map $f^*: C_0 \longrightarrow X - A$ such that $f^*(d) = f(d)$ for all $d \in D_0$.

By our choice of B, there exists a map $F: D_0 \times I \longrightarrow X - A$ and a point $x_0 \in X - A$ such that $F(d, 0) = x_0$ and $F(d, 1) = f(d)$ for all $d \in D_0$. Since A is compact, $X - A$ is an open subset of X, and hence $(X - A) \in$ ANR. By the Borsuk Homotopy Extension Theorem [3, Chapter IV, §8] there exists a map $G: C_0 \times I \longrightarrow X - A$ such that $G(c, 0) = x_0$ for all $c \in C_0$, and $G(d, t) = F(d, t)$ for all $d \in D_0$, $t \in I$. The proof is now completed by defining $f^*: C_0 \longrightarrow X - A$ by $f^*(c) = G(c, 1)$ for all $c \in C_0$.

(3.10) DEFINITION. The proper map $f: X \longrightarrow Y$ *properly factors through the space Z* if there exist proper maps $g: X \longrightarrow Z$ and $h: Z \longrightarrow Y$ such that $f = hg$. If $f: X \longrightarrow Y \subset Y'$ is a proper map, we say that f is *properly nullhomotopic in Y'* if there exists a map $f': X \longrightarrow Y'$ such that $f \simeq_p f'$ in Y' and f' properly factors through a tree. (By *tree* we mean locally finite connected and simply connected 1-complex.) Suppose $X \subset X'$ are spaces. Then X is said to be *properly*

contractible in X' if the inclusion of X into X' is properly nullhomotopic in X'.

(3.11) DEFINITION.   The space X is *properly contractible at* ∞ (X is $PC(E)$) if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that the closure of each component of $X - B$ is properly contractible in $X - A$.

Our immediate goal is to prove that X is $PC(E)$ if $X \in AEC(E)$.   But first we shall require two lemmas.

(3.12) LEMMA.   *If X, $Y \in \Sigma$, $g: X \longrightarrow Y$ is a proper end-preserving map, and B is a compact subset of X such that X is locally connected modulo B, then there exists a compact subset D of Y such that if Y_0 is a component of $Y - D$, then $g^{-1}(Y_0)$ lies in a component of $X - B$.*

PROOF.   Suppose that the hypothesis of the Lemma holds, but that the conclusion fails to hold.   Since $X - B$ is locally connected and $X \in \Sigma$, $X - B$ has only finitely many unbounded components and the union of B and the bounded components of $X - B$ is compact.   (As used here, *bounded* means having compact closure.)   By this latter fact, there exists a compact subset D_1 of Y such that $g^{-1}(D_1)$ contains the union of B and the bounded components of $X - B$. Let $D_1 \subset D_2 \subset \cdots$ be compact subsets of Y such that $Y = \bigcup_{i=1}^{\infty} D_i$.

For $i = 1, 2, \ldots$ there exists a component Y_0^i of $Y - D_i$ such that $g^{-1}(Y_0^i)$ intersects at least two components of $X - B$, and these components of $X - B$ must be unbounded.   We may assume, by selecting a subsequence of $D_1, D_2, \ldots$ if necessary and relabeling, that there exist components U_1 and U_2 of $X - B$ such that $U_1 \cap g^{-1}(Y_0^i) \neq \varnothing \neq U_2 \cap g^{-1}(Y_0^i)$ for $i = 1, 2, \ldots$.   If $i = 1, 2,$ $\ldots$, let $p_i \in U_1 \cap g^{-1}(Y_0^i)$ and $q_i \in U_2 \cap g^{-1}(Y_0^i)$.   Again choosing subsequences and relabeling if necessary, we may assume that $\{p_i\}_{i=1}^{\infty} \longrightarrow e \in EX$ and $\{q_i\}_{i=1}^{\infty} \longrightarrow e' \in EX$.   Since U_1 and U_2 are disjoint open subsets of X having compact frontier, $e \neq e'$.   But $g(p_i)$ and $g(q_i)$ lie in the same component of $Y - D_i$, namely Y_0^i, and hence the sequences $\{g(p_i)\}_{i=1}^{\infty}$ and $\{g(q_i)\}_{i=1}^{\infty}$ have a common limit in EY.   This contradicts the hypothesis that g is end-preserving, completing the proof.

The hypothesis that X be locally connected modulo B is necessary in Lemma 3.12.   To see this, let $X = \bigcup_{i=0}^{\infty} X_i \subset R^2$, where $X_0 = \{(x, y) | 0 \leqslant x \leqslant 1/2, y = 0\}$, $X_1 = \{(x, y) | x = 0, y \geqslant 0\}$,   and $X_n = \{(x, y) | x = 1/n, 0 \leqslant y \leqslant n\}$ for $n = 2, 3, \ldots$.   Let $Y = X_1$ and define $g: X \longrightarrow Y$ by $g(x, y) = (x, 0)$ for all $x \in X$.   Then g is an end-preserving proper map, but if $B = X_0$, D is any compact subset of Y, and Y_0 is the unbounded component of $Y - D$, then $g^{-1}(Y_0)$ will intersect infinitely many components of $X - B$.   Accordingly, to apply Lemma 3.12 to the proof of Theorem 3.14, we shall require the following lemma.   Its proof is routine, and shall be omitted.

(3.13) LEMMA.   *Suppose B is a closed subset of the space X such that for*

any closed set $C \subset I$ and map $f: C \longrightarrow X - B$ carrying C into a component of $X - B$, there exists a map $f^*: I \longrightarrow X$ such that $f^*(c) = f(c)$ for all $c \in C$. Then each component of $X - B$ is locally connected.

Recall that if $f: X \longrightarrow Y$ is a map, then the *mapping cylinder* (M_f) of f is obtained from the disjoint union of $X \times I$ and $Y((X \times I) \oplus Y)$ by identifying $(x, 1) \in X \times I$ with $f(x) \in Y$. Let $\Pi: (X \times I) \oplus Y \longrightarrow M_f$ denote the natural projection. If f is proper, then so is Π, and M_f is locally compact, separable, and metrizable. We follow the usual convention of regarding X and Y as subspaces of M_f, identifying X with $\Pi(X \times \{0\})$ and Y with $\Pi(Y)$.

We are now prepared to proceed to Theorem 3.14.

(3.14) THEOREM. *If $X \in AEC(E)$, then X is $PC(E)$.*

PROOF. Suppose $X \in AEC(E)$, and let A be a compact subset of X. Letting $A = B_0$, inductively choose compact sets $B_0 \subset B_1 \subset \cdots \subset X$ such that $X = \bigcup_{i=1}^{\infty} B_i$ and such that if $i = 1, 2, \ldots, D_0 \subset C_0$ are compacta, and $f: D_0 \longrightarrow X - B_i$ is a map carrying D_0 into a component of $X - B_i$, then there exists a map $f^*: C_0 \longrightarrow X - B_{i-1}$ such that $f^*(d) = f(d)$ for all $d \in D_0$. We may further assume that $B_2 \subset \text{Int } B_3$, so that there exists a compact subset B of X such that $B_2 \subset \text{Int } B \subset B \subset \text{Int } B_3$. Let Y be the closure of one of the components of $X - B$. In order to show that X is $PC(E)$, it will suffice to show that Y is properly contractible in $X - A$. Since this is immediate if Y is compact, we assume henceforth that Y is noncompact.

By [9, Theorem 2.3] there exists a tree T and a proper map $g: Y \longrightarrow T$ such that $ET \cong EY$ and $Fg|EY$ is an embedding. Let M_g denote the mapping cylinder of g and $\Pi: (Y \times I) \oplus T \longrightarrow M_g$ the canonical projection. To show that Y is properly contractible in $X - A$, it clearly suffices to find a proper map $\Phi: M_g \longrightarrow X - A$ such that $\Phi(y) = y$ for all $y \in Y$.

Note that, by Lemma 3.13, if $\hat{Y}$ is the component of $X - B_2$ containing Y, then $\hat{Y}$ is locally connected. Furthermore, since $B \subset \text{Int } B_3 \subset B_3 \subset B_4 \subset \cdots$, $Y - B_j$ is an open subset of $\hat{Y}$ for $j = 3, 4, \ldots$, and thus $Y - B_j$ is locally connected. Also, $B_j \cap Y$ is a compact subset of Y since Y is closed. Hence, we may apply Lemma 3.12 to obtain compact sets $\varnothing = T_0 \subset T_1 \subset \cdots \subset T$ such that $T = \bigcup_{i=0}^{\infty} T_i$ and such that if Z is the closure of one of the components of $T - T_i$, for $i = 0, 1, \ldots$, then $g^{-1}(Z)$ lies in a component of $Y - B_{i+2}$. We may further assume, by "expanding" the T_i's if necessary, that the sets $T_1, T_2, \ldots$ are connected, that $T_i \subset \text{Int } T_{i+1}$, that $T_{i+1} - T_i$ has only finitely many components, and that the components of $T_{i+1} - T_i$ have pairwise disjoint closures in T, for $i = 0, 1, \ldots$. If $i = 1, 2, \ldots$, let K^i denote the closure of $T_i - T_{i-1}$ and let $K_1^i, K_2^i, \ldots, K_{n_i}^i$ denote the components of K^i; let $L^i = g^{-1}(K^i)$, and $L_j^i = g^{-1}(K_j^i)$, where $j = 1, 2, \ldots, n_i$. Then $L_1^i, L_2^i, \ldots, L_{n_i}^i$ are pairwise dis-

joint compacta and each L_j^i lies in a component of $X - B_{i+1}$. For $i = 1, 2,$ $\ldots$, let $P^i = \Pi(L^i \times I) \subset M_g$ and let $P_j^i = \Pi(L_j^i \times I) \subset M_g$ for $j = 1, 2,$ $\ldots, n_i$.

Now $P_j^i \cap Y = L_j^i$ lies in a component of $X - B_{i+1}$, for $i = 1, 2, \ldots$. Hence, if $i = 1, 3, 5, \ldots$, and $j = 1, 2, \ldots, n_i$, there exists a map $\Phi_j^i \colon P_j^i \rightarrow X - B_i$ such that $\Phi_j^i(y) = y$ for all $y \in L_j^i$. Since the sets $P_1^i, P_2^i, \ldots, P_{n_i}^i$ are pairwise disjoint and closed, we may glue the maps $\Phi_1^i, \Phi_2^i, \ldots, \Phi_{n_i}^i$ to obtain a map $\Phi^i \colon P^i \rightarrow X - B_i$ such that $\Phi^i(y) = y$ for all $y \in L^i$.

Now, if $i = 2, 4, 6, \ldots$, and $j = 1, 2, \ldots, n_i$, let

$$\hat{P}_j^i = (P_j^i \cap P^{i-1}) \cup L_j^i \cup (P_j^i \cap P^{i+1}) \subset P_j^i,$$

and define $\Psi_j^i \colon \hat{P}_j^i \rightarrow X - B_{i-1}$ by

$$\Psi_j^i(p) = \begin{cases} \Phi^{i-1}(p) & \text{if } p \in P_j^i \cap P^{i-1}, \\ p & \text{if } p \in L_j^i, \\ \Phi^{i+1}(p) & \text{if } p \in P_j^i \cap P^{i+1}, \end{cases}$$

for all $p \in \hat{P}_j^i$. Then Ψ_j^i is continuous and carries $\hat{P}_j^i$ into a single component of $X - B_{i-1}$. Hence there exists a map $\Phi_j^i \colon P_j^i \rightarrow X - B_{i-2}$ such that $\Phi_j^i(p) = \Psi_j^i(p)$ for all $p \in \hat{P}_j^i$. We now glue the maps $\Phi_1^i, \Phi_2^i, \ldots, \Phi_{n_i}^i$ together to obtain a map $\Phi^i \colon P^i \rightarrow X - B_{i-2}$.

Define $\Phi \colon M_g \rightarrow X - A$ by $\Phi(p) = \Phi^i(p)$ if $p \in P^i$. Then Φ is proper and $\Phi(y) = y$ for all $y \in Y$, so Φ is the map required for the completion of the proof.

If $X \subset X'$ and X is properly contractible in X', then X is contractible in X'. Thus, we have the following

(3.15) THEOREM. *If X is $PC(E)$, then X is $C(E)$.*

Now, combining previous results, we obtain the following equivalence.

(3.16) THEOREM. *If $X \in ANR$, then the following are equivalent.*
  (i) *X is $C(E)$.*
  (ii) *X is $K - C(E)$.*
  (iii) *X is $C - C(E)$.*
  (iv) *$X \in AEC(E)$.*
  (v) *X is $PC(E)$.*

(3.17) DEFINITION. *If $X \in ANR$ and X satisfies (i)–(v) of Theorem 3.16, then X is said to be docile at ∞.*

The following is immediate from Theorem 3.4.

(3.18) THEOREM. *If X, $Y \in$ ANR and $X \leqslant_p Y$, then X is docile at ∞ if Y is docile at ∞.*

In the following section we shall examine the notion of docility as it applies to the Freudenthal compactification of an ANR of the class Σ.

4. The Freudenthal compactification of docile ANR's. In general, it is not the case that $FX \in$ ANR if $X \in$ ANR and $X \in \Sigma$. In this section we shall show that this conclusion can be reached if X is docile at ∞, and we shall show that the converse holds if EX is an unstable subset of FX.

(4.1) LEMMA. *Suppose that $X \in$ ANR, $X \in \Sigma$, and X is docile at ∞. Then if $K \subset L$ are compacta and $g: K \longrightarrow FX$ is a map, g can be extended over a neighborhood U of K in L to a map $h: U \longrightarrow FX$ such that $h^{-1}(EX) = g^{-1}(EX)$.*

PROOF. The idea of the proof is very much the same as that of the proof of Theorem 3.14. Because of this similarity, we take the liberty of avoiding ballooning of notation by assuming that X has precisely one end. The modifications necessary for the general proof will be clear.

Let $B_0 \subset B_1 \subset \cdots$ be compact subsets of X such that $X = \bigcup_{i=0}^{\infty} B_i$, $X - B_i$ is connected for $i = 0, 1, \ldots$, and such that if $D_0 \subset C_0$ are compacta, $j = 1, 2, \ldots$, and $f: D_0 \longrightarrow X - B_j$ is a map, then there exists a map $f^*: C_0 \longrightarrow X - B_{j-1}$ such that $f^*(d) = f(d)$ for all $d \in D_0$. Let $K_0 = g^{-1}(EX)$, and let $N_1, N_2, \ldots$ be compact neighborhoods of K_0 in L such that $K_0 = \bigcap_{i=1}^{\infty} N_i$, $N_{i+1} \subset \text{Int } N_i$, and $g(K \cap N_i) \subset FX - B_{i+1}$ for $i = 1, 2, \ldots$. Let $M_i = \text{Cl}(N_i - N_{i+1})$ for $i = 1, 2, \ldots$, and let 　　　　if $n \in K_0$,

For every odd positive integer i, let $\Phi_i: M_i \longrightarrow X - B_i$ be a map such that $\Phi_i(m) = g(m)$ for all $m \in M_i'$. Now, fixing i to be an even positive integer, let $M_i'' = (M_i \cap M_{i-1}) \cup M_i' \cup (M_i \cap M_{i+1})$, and define $\Psi_i: M_i'' \longrightarrow X - B_{i-1}$ by

$$\Psi_i(m) = \begin{cases} \Phi_{i-1}(m) & \text{if } m \in M_i \cap M_{i-1}, \\ g(m) & \text{if } m \in M_i', \\ \Phi_{i+1}(m) & \text{if } m \in M_i \cap M_{i+1}, \end{cases}$$

for all $m \in M_i''$. There exists a map $\Phi_i: M_i \dashrightarrow X - B_{i-2}$ such that $\Phi_i(m) = \Psi_i(m)$ for all $m \in M_i''$. Now, define the map $h': N_1 \longrightarrow FX$ by

$$h'(n) = \begin{cases} \Phi_i(n) & \text{if } n \in M_i, \\ g(n) & \text{if } n \in K_0, \end{cases}$$

for all $n \in N_1$. Let $Z = L - \text{Int } N_1$, let $Z_0 = (K \cap Z) \cup \text{Fr } N_1$, and define $h'': Z_0 \longrightarrow X$ by

$$h''(z) = \begin{cases} g(z) & \text{if } z \in K \cap Z, \\ h'(z) & \text{if } z \in \text{Fr } N_1, \end{cases}$$

for all $z \in Z_0$. Then, since Z_0 is closed in Z and $X \in$ ANR, there exist a neighborhood U_0 of Z_0 in Z and a map $h^*\colon U_0 \longrightarrow X$ such that $h^*(z) = h''(z)$ for all $z \in Z_0$. Let $U = U_0 \cup N_1$. Then U is a neighborhood of K in L, and the required map $h\colon U \longrightarrow FX$ may be defined by

$$h(u) = \begin{cases} h^*(u) & \text{if } u \in U_0, \\ h'(u) & \text{if } u \in N_1, \end{cases}$$

for all $u \in U$.

If Z is a space and $Z' \subset Z$, then Z' is an *unstable* subset of Z if there exists a homotopy $H\colon Z \times I \longrightarrow Z$ such that $H(z, 0) = z$ and $H(z, t) \notin Z'$ for all $z \in Z$ and $0 < t \leqslant 1$.

The following generalizes a result of Kozlowski [7].

(4.2) THEOREM. *If $X \in$ ANR and $X \in \Sigma$, then X is docile at ∞ if and only if $FX \in$ ANR and EX is an unstable subset of FX.*

PROOF. Suppose first that $X \in$ ANR, $X \in \Sigma$, and X is docile at ∞. By Lemma 4.1, FX is an absolute neighborhood extensor for compact metric spaces, and hence $FX \in$ ANR. It remains to be shown that EX is an unstable subset of FX. Let $K = FX \times \{0\} \subset FX \times I = L$, and define $g\colon K \longrightarrow FX$ by $g(x, 0) = x$ for all $x \in FX$. By Lemma 4.1, there exists a neighborhood U of K in L and a map $h\colon U \longrightarrow FX$ such that $h(x, 0) = x$ for all $x \in FX$ and $h^{-1}(EX) = g^{-1}(EX) = EX \times \{0\}$. Choose $t_0 \in (0, 1]$ such that $FX \times [0, t_0] \subset U$, and define $\Phi\colon FX \times I \longrightarrow FX$ by $\Phi(x, t) = h(x, tt_0)$ for all $(x, t) \in FX \times I$. Then $\Phi(x, 0) = x$ for all $x \in FX$ and, if $t > 0$, $\Phi(x, t) \in FX - EX$. This shows that EX is an unstable subset of FX.

For the converse, suppose now that $FX \in$ ANR and that EX is an unstable subset of FX. Let $\Phi\colon FX \times I \longrightarrow FX$ be a homotopy such that $\Phi(x, 0) = x$ for all $x \in FX$ and $\Phi(x, t) \notin EX$ for all $x \in FX$ and $t \in (0, 1]$. To show that X is docile at ∞, it suffices to show that X is $C - C(E)$. To this end, let A be a compact subset of X, and let A' be a compact subset of X such that $A \subset \text{Int } A'$. Since $FX \in$ ANR, FX is LC at e for all $e \in EX$. It follows that there exist open sets $U_1, U_2, \ldots, U_n$ in FX covering EX such that U_i is contractible in $FX - A'$ for $i = 1, 2, \ldots, n$. Let B be a compact subset of X such that each component of $X - B$ lies in one of the sets $U_1, U_2, \ldots, U_n$. We will show that any map of a compactum into a component of $X - B$ is nullhomotopic in $X - A$, thereby completing the proof.

Let C_0 be a compactum and $f\colon C_0 \longrightarrow X - B$ a map carrying C_0 into a

component of $X - B$. Then $f(C_0) \subset U_j$ for some j. Hence there exists a map $F: C_0 \times I \longrightarrow FX - A'$ and a point $p_0 \in FX$ such that $F(c, 0) = f(c)$ and $F(c, 1) = p_0$ for all $c \in C_0$. Since $A \cap \mathrm{Cl}(FX - A') = \emptyset$, we may choose $t_0 \in (0, 1]$ such that $\Phi(\mathrm{Cl}(FX - A') \times [0, t_0]) \subset FX - A$. Now define $G: C_0 \times I \longrightarrow X - A$ by $G(c, t) = \Phi(F(c, t), tt_0)$ for all $(c, t) \in C_0 \times I$. Then G is a homotopy joining f with a constant map in $X - A$, and the proof is complete.

(4.3) LEMMA. *If Z' is an unstable subset of the space Z, then the inclusion of $Z - Z'$ into Z is a homotopy equivalence.*

PROOF. Let $H: Z \times I \longrightarrow Z$ be a map such that $H(z, 0) = z$ and $H(z, t) \notin Z'$ for all $z \in Z$ and $0 < t \leqslant 1$. Let $i: Z - Z' \longrightarrow Z$ be the inclusion and define $f: Z \longrightarrow Z - Z'$ by $f(z) = H(z, 1)$ for all $z \in Z$. Then, defining $F_1: (Z - Z') \times I \longrightarrow Z - Z'$ by $F_1(z, t) = H(z, t)$ for all $z \in Z - Z'$ and $t \in I$, we see that F_1 is a homotopy joining $1_{Z-Z'}$ and fi. Since H is a homotopy joining 1_Z and if, the proof is complete.

The following is immediate from Lemma 4.3 and Theorem 4.2.

(4.4) THEOREM. *If $X \in ANR$, $X \in \Sigma$, and X is docile at ∞, then the inclusion of X into FX is a homotopy equivalence.*

In [10], the author introduced the notion of *absolute proper retract (APR)*. The next result shows how our current work relates to this topic. We use the fact [10, Theorem 4.2] that $X \in APR$ if and only if X is noncompact, $FX \in AR$, and EX is an unstable subset of FX.

(4.5) THEOREM. *Suppose $X \in ANR$. Then $X \in APR$ if and only if X is noncompact, contractible, and docile at ∞.*

PROOF. If $X \in APR$, then X is a noncompact AR [10]. Hence X is noncompact and contractible. Furthermore, $FX \in AR$ and EX is an unstable subset of FX so, by Theorem 4.2, X is docile at ∞. Conversely, if X is noncompact, contractible, and docile at ∞, then $FX \in ANR$ and EX is an unstable subset of FX. Furthermore, by Theorem 4.4, the inclusion of X into FX is a homotopy equivalence, so FX is contractible, hence an AR. By Theorem 2.1, $X \in APR$.

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